

Unified Engineering Problem Set
Week 14 Fall, 2007

SOLUTIONS

M14.1 Compliance tensor

(a) Start with:

$$\epsilon_{mn} = S_{mnpq} \sigma_{pq}$$

This has the same form as the elasticity equation:

$$\sigma_{mn} = E_{mnpq} \epsilon_{pq}$$

and since the same symmetries exist for the compliance tensor as for the elasticity tensor, the full anisotropic equations have the same form.

Thus:

$$\epsilon_{11} = S_{1111} \sigma_{11} + S_{1122} \sigma_{22} + S_{1133} \sigma_{33} \\ + 2S_{1123} \sigma_{23} + 2S_{1113} \sigma_{13} + 2S_{1112} \sigma_{12}$$

$$\epsilon_{22} = S_{1122} \sigma_{11} + S_{2222} \sigma_{22} + S_{2233} \sigma_{33} \\ + 2S_{2223} \sigma_{23} + 2S_{2213} \sigma_{13} + 2S_{2212} \sigma_{12}$$

(cont.)

(cont. from previous page)

$$\epsilon_{33} = S_{1133} \sigma_{11} + S_{2233} \sigma_{22} + S_{3333} \sigma_{33} \\ + 2S_{3323} \sigma_{23} + 2S_{3313} \sigma_{13} + 2S_{3312} \sigma_{12}$$

$$\epsilon_{23} = S_{1123} \sigma_{11} + S_{2223} \sigma_{22} + S_{3323} \sigma_{33} \\ + 2S_{2323} \sigma_{23} + 2S_{2313} \sigma_{13} + 2S_{2312} \sigma_{12}$$

$$\epsilon_{13} = S_{1113} \sigma_{11} + S_{2213} \sigma_{22} + S_{3313} \sigma_{33} \\ + 2S_{1323} \sigma_{23} + 2S_{1313} \sigma_{13} + 2S_{1312} \sigma_{12}$$

$$\epsilon_{12} = S_{1112} \sigma_{11} + S_{2212} \sigma_{22} + S_{3312} \sigma_{33} \\ + 2S_{1223} \sigma_{23} + 2S_{1213} \sigma_{13} + 2S_{1212} \sigma_{12}$$

Likewise, these can be grouped into the 3 groups similar to that for the elasticity tensor, \bar{E}_{unip} :

$$\begin{array}{l} \text{extensional stresses} \\ \text{to} \\ \text{extensional strains} \end{array} \left\{ \begin{array}{ll} S_{1111} & S_{1122} \\ S_{2222} & S_{1133} \\ S_{3333} & S_{2233} \end{array} \right.$$

$$\begin{array}{l} \text{shear stresses} \\ \text{to} \\ \text{shear strains} \end{array} \left\{ \begin{array}{ll} S_{1212} & S_{1213} \\ S_{1313} & S_{1323} \\ S_{2323} & S_{1223} \end{array} \right.$$

COUPLING TERMS

extensional stresses to
shear strain

or
shear stresses to
extensional strains

$$\left\{ \begin{array}{lll} S_{1112} & S_{2212} & S_{3312} \\ S_{1113} & S_{2213} & S_{3313} \\ S_{1123} & S_{2223} & S_{3323} \end{array} \right.$$

(b) For the orthotropic case, all COUPLING TERMS are zero (in the principal axes of the material). Thus:

$$\begin{array}{lll} S_{1112} = 0 & S_{2212} = 0 & S_{3312} = 0 \\ S_{1113} = 0 & S_{2213} = 0 & S_{3313} = 0 \\ S_{1123} = 0 & S_{2223} = 0 & S_{3323} = 0 \end{array}$$

In addition, shear stresses (strains) in one plane do not cause shear strain (stresses) in another plane. Thus, three of the shear stress to shear strain terms become zero:

$$S_{1213} = 0$$

$$S_{1323} = 0$$

$$S_{1223} = 0$$

All other terms are nonzero and independent.

This gives 9 independent compliance components as for the elasticity case. The resulting equations are:

$$\epsilon_{11} = S_{1111} \sigma_{11} + S_{1122} \sigma_{22} + S_{1133} \sigma_{33}$$

$$\epsilon_{22} = S_{1122} \sigma_{11} + S_{2222} \sigma_{22} + S_{2233} \sigma_{33}$$

$$\epsilon_{33} = S_{1133} \sigma_{11} + S_{2233} \sigma_{22} + S_{3333} \sigma_{33}$$

$$\epsilon_{23} = 2S_{2323} \sigma_{23}$$

$$\epsilon_{13} = 2S_{1313} \sigma_{13}$$

$$\epsilon_{12} = 2S_{1212} \sigma_{12}$$

From the lecture notes, (M 3.2, p. 29) the stress-strain relations for the orthotropic case using engineering constants are:

$$\epsilon_1 = \frac{1}{E_1} [\sigma_1 - \nu_{12} \sigma_2 - \nu_{13} \sigma_3]$$

$$\epsilon_2 = \frac{1}{E_2} [-\nu_{21} \sigma_1 + \sigma_2 - \nu_{23} \sigma_3]$$

$$\epsilon_3 = \frac{1}{E_3} [-\nu_{31} \sigma_1 - \nu_{32} \sigma_2 + \sigma_3]$$

$$\gamma_{23} = \frac{1}{G_{23}} \sigma_{23}$$

$$\gamma_{13} = \frac{1}{G_{13}} \sigma_{13}$$

$$\gamma_{12} = \frac{1}{G_{12}} \sigma_{12}$$

To go from tensorial to engineering notation for stress and strain, note that:

$$\begin{array}{ll}
 \epsilon_{11} = \epsilon_1 & \sigma_{11} = \sigma_1 \\
 \epsilon_{22} = \epsilon_2 & \sigma_{22} = \sigma_2 \\
 \epsilon_{33} = \epsilon_3 & \sigma_{33} = \sigma_3 \\
 2\epsilon_{23} = \gamma_{23} & \sigma_{23} = \sigma_{23} \\
 2\epsilon_{13} = \gamma_{13} & \sigma_{13} = \sigma_{13} \\
 2\epsilon_{12} = \gamma_{12} & \sigma_{12} = \sigma_{12}
 \end{array}$$

Using these relations with the previous two sets of equations results in:

$$\begin{array}{l}
 S_{1111} = 1/E_1 \\
 S_{1122} = -\nu_{12}/E_1 = -\nu_{21}/E_2 \quad \leftarrow \text{via reciprocity} \\
 S_{1133} = -\nu_{13}/E_1 = -\nu_{31}/E_3 \quad \leftarrow \text{via reciprocity} \\
 S_{2222} = 1/E_2 \\
 S_{2233} = -\nu_{23}/E_2 = -\nu_{32}/E_3 \quad \leftarrow \text{via reciprocity} \\
 S_{3333} = 1/E_3 \\
 S_{2323} = 1/(4G_{23}) \\
 S_{1313} = 1/(4G_{13}) \\
 S_{1212} = 1/(4G_{12})
 \end{array}
 \quad \left. \vphantom{\begin{array}{l} S_{1111} \\ S_{1122} \\ S_{1133} \\ S_{2222} \\ S_{2233} \\ S_{3333} \\ S_{2323} \\ S_{1313} \\ S_{1212} \end{array}} \right\} (*)$$

(*) NOFF factor of 4 in these cases.

Look at one particular case:

$$\epsilon_{23} = 2S_{2323} \sigma_{23} \text{ and } \gamma_{23} = \frac{1}{G_{23}} \sigma_{23}$$

$$\text{along with } 2\epsilon_{23} = \gamma_{23}$$

$$\Rightarrow 2\epsilon_{23} = 4S_{2323} \sigma_{23} = \gamma_{23} = \frac{1}{G_{23}} \sigma_{23}$$

$$\Rightarrow 4S_{2323} = \frac{1}{G_{23}}$$

Finally: $S_{2323} = \frac{1}{(4G_{23})}$

..... same for other two cases

(c) we know that the compliance matrix is the inverse of the elasticity matrix or vice versa:

$$\underline{\underline{E}} = \underline{\underline{S}}^{-1}$$

for this orthotropic case, the compliance matrix is:

$$\underline{\underline{S}} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{1122} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{1133} & S_{2233} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2S_{1212} \end{bmatrix}$$

We also found the relations between the components of the compliance tensor and the engineering constants. We can use these in the compliance matrix:

$$\underline{S} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & -\nu_{13}/E_1 & 0 & 0 & 0 \\ -\nu_{21}/E_2 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0 \\ -\nu_{31}/E_3 & -\nu_{32}/E_3 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(2G_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/(2G_{13}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(2G_{12}) \end{bmatrix}$$

(eq. 1)

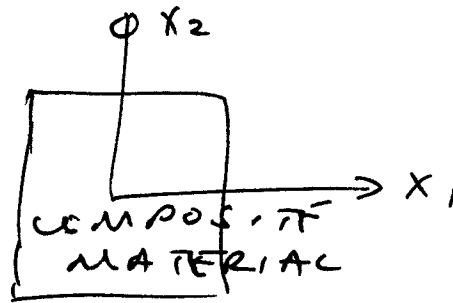
→ For the orthotropic case, the elasticity matrix is:

$$\underline{E} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E_{1212} \end{bmatrix}$$

(eq. 2)

Now take the inverse of the matrix expression of (1) and equate it to the matrix expression of (2). This will give relations between the components of the elasticity tensor and the engineering constants.

M14.2



Experiment A: $\sigma_{11} = 200 \text{ MPa}$
 $\sigma_{22} = 200 \text{ MPa}$
 $\epsilon_{22} = 3500 \text{ } \mu\text{strain}$

Experiment B: $\sigma_{11} = 600 \text{ MPa}$
 $\epsilon_{11} = 7800 \text{ } \mu\text{strain}$
 $\epsilon_{22} = -2250 \text{ } \mu\text{strain}$

Experiment C: $\sigma_{12} = 150 \text{ MPa}$
 $\epsilon_{12} = 3400 \text{ } \mu\text{strain}$

Stresses and strains not specified are zero (except ϵ_{11} for Experiment A as the page broke)

(a) Experiments A and B show that extensional stresses cause only extensional strains. Experiment C shows that shear stress causes only shear strain. Thus, there is no shear-extensional coupling and

this behavior behaves "at most" as an orthotropic material.

Since the out-of-plane stresses (σ_{3j}) are also zero in all three experiments, we only need to use the in-plane form of the stress-strain equations. This gives:

$$\epsilon_{11} = \frac{1}{E_1} \sigma_{11} - \frac{\nu_{21}}{E_2} \sigma_{22} \quad (1)$$

$$\epsilon_{22} = -\frac{\nu_{12}}{E_1} \sigma_{11} + \frac{1}{E_2} \sigma_{22} \quad (2)$$

$$2\epsilon_{12} = \frac{1}{G_{12}} \sigma_{12} \quad (3)$$

→ Using the results of Experiment C in (3):

$$2(3400 \times 10^{-6}) = \frac{1}{G_{12}} (150 \times 10^6 \text{ Pa})$$

$$\Rightarrow G_{12} = 22.1 \times 10^9 \text{ Pa} = 22.1 \text{ GPa}$$

→ Use the results of Experiment B (since there is only one stress applied) in (1):

$$7800 \times 10^{-6} = \frac{1}{E_1} (600 \times 10^6 \text{ Pa})$$

$$\Rightarrow E_1 = 76.9 \times 10^9 \text{ Pa} = 76.9 \text{ GPa}$$

and then in (2) along with the value determined for E_1 :

$$-2250 \times 10^{-6} = \frac{-\nu_{12}}{(76.9 \times 10^9 \text{ Pa})} (600 \times 10^6 \text{ Pa})$$

$$\Rightarrow \boxed{\nu_{12} = 0.288}$$

→ Use the results of Experiment A in (2) along with the results for ν_{12} and E_1 , as determined via the results of Experiment B:

$$3500 \times 10^{-6} = \frac{-0.288}{(76.9 \times 10^9 \text{ Pa})} (200 \times 10^6 \text{ Pa}) + \frac{1}{E_2} (200 \times 10^6 \text{ Pa})$$

$$\Rightarrow 3500 \times 10^{-6} = -750 \times 10^{-6} + \frac{1}{E_2} (200 \times 10^6 \text{ Pa})$$

$$\Rightarrow 4250 \times 10^{-6} = \frac{1}{E_2} (200 \times 10^6 \text{ Pa})$$

$$\Rightarrow \boxed{E_2 = 47.1 \times 10^9 \text{ Pa} = 47.1 \text{ GPa}}$$

→ One can also use the reciprocity relation:

$$\nu_{21} E_1 = \nu_{12} E_2$$

and the values determined to get ν_{21} :

$$\nu_{21} (76.9 \text{ GPa}) = (0.288) (47.16 \text{ Pa})$$

$$\Rightarrow \boxed{\nu_{21} = 0.176}$$

This gives the in-plane engineering constants for this orthotropic material.

With these 5 constants (4 independent) and the equations with which we started, the in-plane stress-strain behavior of the material is characterized.

(b) Use the results from Experiment A and the results from (a) to determine ϵ_{11} for that Experiment, if possible.

It is possible and the key is in the use of the reciprocity relation to determine a value for ν_{21} . Thus, all the terms in equation (1) are determined and for Experiment A, one can determine:

$$\epsilon_{11} = \frac{1}{E_1} \sigma_{11} - \frac{\nu_{21}}{E_2} \sigma_{22}$$

$$\Rightarrow \epsilon_{11} = \frac{1}{(76.9 \times 10^9 \text{ Pa})} (200 \times 10^6 \text{ Pa}) - \frac{0.176}{(47.1 \times 10^9 \text{ Pa})} (200 \times 10^6 \text{ Pa})$$

$$\Rightarrow \epsilon_{11} = 2601 \times 10^{-6} - 747 \times 10^{-6}$$

$$\Rightarrow \boxed{\epsilon_{11} = 1854 \times 10^{-6} = 1854 \mu\text{strain}}$$

(c) Begin by using the stress-strain equations in matrix form:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/2G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Now consider the compliance tensor relations also in matrix form and with the zero stresses ($\sigma_{13}, \sigma_{23}, \sigma_{33}$) and strains ($\epsilon_{13}, \epsilon_{23}, \epsilon_{33}$) ignored/eliminated:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & 2S_{1112} \\ S_{2211} & S_{2222} & 2S_{2212} \\ S_{1211} & S_{1222} & 2S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

We know the COUPLING TERMS are zero based on the experiments showing this to be orthotropic, so we are left with 5 terms.

We compare these directly with those from the matrix expression using engineering constants and find:

$$S_{1111} = 1/E_1 = \frac{1}{76.9 \times 10^9 \text{ Pa}} = 1.30 \times 10^{-11} \frac{1}{\text{Pa}}$$

$$S_{1122} = S_{2211} = \frac{-\nu_{21}}{E_2} = \frac{-\nu_{12}}{E_1} = \frac{-0.288}{76.9 \times 10^9 \text{ Pa}} = -0.375 \times 10^{-11} \frac{1}{\text{Pa}}$$

$$S_{2222} = 1/E_2 = \frac{1}{47.1 \times 10^9 \text{ Pa}} = 2.12 \times 10^{-11} \frac{1}{\text{Pa}}$$

$$2S_{1212} = 2G_{12} = \frac{1}{2(22.1 \times 10^9 \text{ Pa})} = 2.26 \times 10^{-11} \frac{1}{\text{Pa}}$$

$$\Rightarrow S_{1212} = 1.13 \times 10^{-11} \frac{1}{\text{Pa}}$$

Summarizing:

$$\begin{array}{l} S_{1111} = 1.30 \times 10^{-11} \frac{1}{\text{Pa}} \quad S_{1122} = S_{2211} = -0.375 \times 10^{-11} \frac{1}{\text{Pa}} \\ S_{2222} = 2.12 \times 10^{-11} \frac{1}{\text{Pa}} \quad S_{1212} = 1.13 \times 10^{-11} \frac{1}{\text{Pa}} \\ S_{1112} = S_{2212} = S_{1211} = S_{1222} = 0 \end{array}$$

also shown in experiment

And in matrix form:

$$\underline{\underline{S}} = \begin{bmatrix} 1.30 & -0.375 & 0 \\ -0.375 & 2.12 & 0 \\ 0 & 0 & 2(1.13) \end{bmatrix} \times \left[10^{-11} \frac{1}{\text{Pa}} \right]$$

$$\text{for: } \underline{\underline{\epsilon}} = \underline{\underline{S}} \underline{\underline{\sigma}}$$